und Dichtewerten zu erwartende Rate, die etwa um einen Faktor 2,7 höher ist. Dieser quantitative Unterschied ist gut verständlich im Hinblick auf die angegebenen Fehlergrenzen und im Hinblick auf die zur Zeit noch nicht sehr sichere Kenntnis von Anregungsquerschnitten, die in die Rechnungen eingehen; zudem haben die Anregungsstöße der hier zusätzlich anwesenden energiereichen Elektronen und die weitgehende optische Dicke der LYMAN-Strahlung die Tendenz, die Rekombinationsrate etwas zu erniedrigen.

Für die Unterstützung dieser Untersuchungen bin ich sehr dankbar dem Max-Planck-Institut für Physik und Astrophysik, München, und der National Science Foundation.

On Drift Instabilities in a Collisionless Electron-Proton Plasma from the point of view of Energy Absorption of Resonant Particles

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(Z. Naturforschg. **18 a**, 446—453 [1963]; eingegangen am 10. September 1962)

The present paper is essentially devoted to the study of instabilities of electrostatic waves in a current-carrying collisionless plasma. As the underlying physical cause of the instabilities is the same as that of the LANDAU damping in an electron plasma, a detailed analysis of the latter is first given. It is shown that the damping may be considered as being due to the fact that there are more electrons in the phase-region where energy is absorbed by the particles from the field than in the phase-region where energy is given up to the field.

We then proceed to the evaluation of the energy absorption A of the resonant particles, first in the absence of an external magnet field, $\boldsymbol{B}_0$, next when the wave is propagated under an arbitrary angle with respect to $\boldsymbol{B}_0$. When $A > 0$, the wave is damped, and vice-versa. Without appeal to a dispersion equation, stability criteria can thus be found, dependent on the wave frequency and wave-vector. Next some special cases are investigated and compared with the results of other authors where such results exist.

As a consequence of the fact that some ions and electrons, the resonant particles, experience a constant electric field, these particles also experience a constant drift transverse to both $\boldsymbol{E}$ and $\boldsymbol{B}_0$. This drift gives rise to a transverse current which is closely related to the damping or growing of the wave. An expression for this current, averaged over one wave-length is found.

I. Introduction

In the latest years many authors have been investigating the stability of two interpenetrating components of a plasma when particle-collisions are neglected. This phenomenon is of considerable interest in the study of ionized gas discharges, klystrons, throchotrons and perhaps the most important, in the study of different fusion experiments.

The first theoretical investigation of instabilities of this kind was done by PIERCE [1] and HAEFF [2]. They showed that two interpenetrating streams with no thermal motions can be unstable. A physical explanation of this kind of instability was given by BOHM and GROSS [3].

Later BUNEMAN [4, 5] studied two-stream instabilities thoroughly and found some stability criteria.

When thermal motions are taken into account, the phenomena become more complicated, and the physical cause of the instability or damping is more obscure.

To solve the adequate dispersion equation for the problem, different approximations must be made. JACKSON [6] and PENROSE [7] employed the NYQUIST criterium and were able to find some general criteria for stability of a wave propagating in a collisionless many component plasma without external fields.

While PENROSE used non-Maxwellian distribution functions for the different plasma constituents, A. JACKSON [8] studied the stability of two Maxwellian

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components of a plasma with different drift velocities. JACKSON used a graphical technique and gave a detailed analysis of how the instability depends on the wavelength, the difference in drift velocity of the two components, the ratio of the DEBYE lengths and particle masses.

Animated by some experimental results from the B-3 Stellarator, BERNSTEIN, FRIEMAN, KULSRUD and ROSENBLUTH [9] in a short note, investigated the wave instability in a collisionless two component plasma embedded in an external magnetic field. As the wave frequency was rather low, the instability was essentially that of an ion wave.

The authors showed that these ion wave instabilities could lead to a transport of plasma across the magnetic field and were thus able to give a tentative explanation of the observed particle loss in the Project Matterhorn B-3 Stellarator (SPITZER [10]). (Recent experiments, STODICK et al. [11] and MOTLEY [12] indicate, however, that the loss of particles cannot be explained in full as being due to ion wave instabilities.)

Later BERNSTEIN and KULSRUD [13–15] gave a broader treatment of ion wave instabilities, deriving dispersion equations which they solved for different limiting cases.

It is shown that when the relative velocity between ions and electrons is large enough, ion wave instabilities occur.

Further investigation was undertaken by SUMI [16] which studied plasma oscillations in an external magnetic field and an external electric field. A drift is thus produced by the latter field and it is in addition time-dependent. The strength of the electric field is, however, assumed very small in SUMI's work.

Many other papers on drift instabilities have appeard. The above "list" is thus not complete but is fairly representative as a background for the present investigation.

In this paper drift instabilities will be studied from a physical point of view, namely by calculating the transfer of energy from the particles to the wave (instability), or from the wave to the particles (stability). When there is no transfer, the wave is a neutral one.

Without appeal to the dispersion equation, preliminary stability criteria (criteria where the real wave frequency and wavenumber appear) will be found. The calculations are an extension of previous works by the writer, KILDAL [17–19] where the LANDAU damping was calculated from energy considerations. As the same considerations now are applied to the drift stabilities or instabilities, the latter may thus be caracterised as a generalized LANDAU damping or growing. The origin of the phenomenon is in any case a resonance effect, as particles with a certain velocity experience a constant electric field when moving in the wave.

As we lean so heavily on the physical mechanism of LANDAU damping, we shall present some comments on the latter in addition to what is said in the previous papers.

II. The Physical Mechanism of Landau Damping

The correct expression for the damping in the non-magnetic, long wave-length case,

$$\gamma = -\frac{\pi\,\omega_p^2}{2\,k^2\,N}\,\omega\left(1-\frac{k}{\omega}\frac{d\omega}{dk}\right) f_0'\left(\frac{\omega}{k}\right), \tag{2.1}$$

has been calculated by DAWSON [20] and by KILDAL from a physical point of view.

In (2.1) ω = wave frequency, k = wave number, ω_p = plasma frequency, N = electron density and $f_0(v)$ = electron distribution function. The physical conditions are an unbounded electron plasma neutralized by motionless ions and no particle collisions.

In order to arrive at (2.1) valid for both $\gamma > 0$ (damping) and $\gamma < 0$ (growing) an analytic continuation had to be performed as the expression derived was valid only for $\gamma < 0$. See KILDAL [19] (hereafter Paper III), where also the physical meaning of the continuation is recorded.

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We see from (2.1) that when $f_0'(\omega/k) < 0$ the wave is damped and when $f_0'(\omega/k) > 0$ it is a growing wave. We shall now show that the physical content of this fact is the following. In the case of instability, the density of particles with velocity ω/k is larger in a phase region where energy is given up to the field than in a region where energy is absorbed by the particles from the field. For damping the contrary applies.

To demonstrate this, some results from Paper III will be borrowed.

We let the electric field vary as

$$E = E_0\, e^{i(kx-\omega' t)} = E_0\, e^{-\gamma t}\, e^{i(kx-\omega t)}, \quad (2.2)$$

where $\omega' = \omega - i\gamma$ is the complex frequency. The perturbation f_{1e} of the distribution function $f_{0e}(v)$ is then found to be

$$f_{1e}(x, v, t) = -i\,\frac{e\,E}{m}\, e^{-\gamma t}\, \frac{f'_{0e}(v)}{k\,v-\omega'}, \quad (2.3)$$

the real part of which is

$$f_{1e}(x, v, t) = \frac{e\,E_0}{m}\, e^{-\gamma t} f'_{0e}(v) \cdot \frac{-\gamma\cos(k\,x-\omega\,t) + (k\,v-\omega)\sin(k\,x-\omega\,t)}{(k\,v-\omega)^2+\gamma^2}. \quad (2.4)$$

Here m = electron mass.

The energy absorbed by the resonant electrons is given by

$$A = -e \int_{-\infty}^{\infty} \overline{v\,E(f_{0e}(v) + f_{1e})}\, dv \quad (2.5)$$

when calculated to zero order in γ[21]. The bar indicates mean value in space, i. e., over one wavelength.

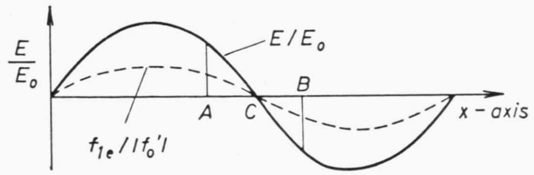
Now since v and $f_{0e}(v)$ are independent of x, the mean value of $v\,E\,f_{0e}$ is zero, and the only contribution to (2.5) comes from $v \cdot \overline{E\,f_{1e}}$. From (2.4) we see that f_{1e} has one component f''_{1e} proportional to $\sin(k\,x-\omega\,t)$ and another component f'_{1e} which is proportional to $\cos(k\,x-\omega\,t)$. Only the latter contributes to $\overline{E\,f_{1e}}$.

From (2.4) we see that for particles with a velocity v essentially different from ω/k, i. e. $|k\,v-\omega| \gg \gamma$, the function $f''_{1e} \gg f'_{1e}$. For particles such that $v \approx \omega/k$, $f''_{1e} \ll f'_{1e}$. As it is the latter group, the resonant electrons, which contributes to A, in the case $\gamma \to 0$, we shall neglect f''_{1e}.

[21] In this paper and in other papers of the writer referred to, A is always calculated to zero order in γ, which physically means that only the resonant absorption is considered.

Then E and $f_{1e}(v \approx \omega/k)$ are in phase. Let us now assume a growing wave $\gamma < 0$, i. e. $f_0'(\omega/k) > 0$.

Looking at Fig. 1 this means that there are more electrons at A than at B where A and B are two points in the wave, symmetric with respect to a



point C where $E = 0$. The particles at A give up energy to the electric field while those at B absorb energy from the field. As the number of electrons at A is larger than the number at B, in the mean over λ energy is absorbed by the field and the latter grows. The increment factor $-\gamma$ is found from the expression

$$2\,\gamma = A/W, \quad (2.6)$$

where

$$W = \frac{E_0^2}{8\,\pi}\, e^{-2\gamma t} \cdot \left(1 - \frac{k}{\omega}\,\frac{d\omega}{dk}\right)^{-1}, \quad (2.7)$$

is the energy density in the wave in the laboratory frame.

The same considerations may be applied to the case $\gamma > 0$.

III. Drift Instabilities without an External Magnetic Field

We shall now use the same method as in our previous papers to calculate the damping when both electrons and ions are taking part in the motion. We shall further assume that there is a displacement u between the points where $f_{0e}(v)$ and $f_{0i}(v)$ have their maxima and consequently there is a current in the plasma. The formula for the energy absorption is

$$A = -e\int_{-\infty}^{\infty} v\,\overline{E\,f_{1e}}\, dv + e\int_{-\infty}^{\infty} v\,\overline{E\,f_{1i}}\, dv, \quad (3.1)$$

where the bar indicates meanvalue in space. Substituting from eq. (2.4) and the analogous expression for f_{1i}, the ion distribution function, we get:

$$A = -\frac{1}{2} e^2 E_0^2 e^{-2\gamma t} \int_{-\infty}^{\infty} v \frac{-\gamma[(1/m) f'_{0e}(v) + (1/M) f'_{0i}(v)] \, dv}{(k v - \omega)^2 + \gamma^2}. \tag{3.2}$$

Now, in the limit $\gamma/k \to -0$, we obtain $A = \frac{-\omega_{pe}^2 \omega}{8 k^2 N} E_0^2 e^{-2\gamma t} \left[f'_{0e}(v) + \frac{m}{M} f'_{0i}(v) \right].$ (3.3)

The energy density of the wave being given by (2.7), the imaginary part of the frequency becomes

$$\gamma = \frac{-\pi \omega_{pe}^2 \omega}{2 k^2 N} \left(1 - \frac{k}{\omega} \frac{d\omega}{dk} \right) \left[f'_{0e}\left(\frac{\omega}{k}\right) + \frac{m}{M} f'_{0i}\left(\frac{\omega}{k}\right) \right]. \tag{3.4}$$

A necessary criterium for instability is thus $f'_{0e}\left(\frac{\omega}{k}\right) + \frac{m}{M} f'_{0i}\left(\frac{\omega}{k}\right) > 0,$ (3.5)

a condition which agrees with the result of the authors referred to which have done similar calculations based on the dispersion equation.

Let us now see what the physical meaning of condition (3.5) is, remembering that we are studying the case $\gamma/\omega \ll 1$. We have in view of what is said in section 1, that f_{1j} is essentially given by

$$f_{1j}(x, v \approx \omega/k, t) = -\frac{q_j E_0}{m_j} f'_{0j}(v) \, e^{-\gamma t} \frac{-\gamma \cos(k x - \omega t)}{(k v - \omega)^2 + \gamma^2} \tag{3.6}$$

for the resonant particles. (j = e, i and $q_e = -e$, $q_i = e$).

With the distribution functions given by

$$f_{0e}(v) = N \left(\frac{m}{2 \pi \varkappa T_e} \right)^{1/2} \exp\left\{ -\frac{m (v-u)^2}{2 \varkappa T_e} \right\}, \qquad f_{0i}(v) = N \left(\frac{M}{2 \pi \varkappa T_i} \right)^{1/2} \exp\left\{ -\frac{M v^2}{2 \varkappa T_i} \right\}$$

we see that $f'_{0e}(\omega/k) > 0$ and $f'_{0i}(\omega/k) < 0$, and the condition (3.5) is essentially

$$\frac{1}{E} [f_{1e}(x, \omega/k, t) - f_{1i}(x, \omega/k, t)] > 0 \quad \text{or} \quad |f_{1e}(x, \omega/k, t)| > |f_{1i}(x, \omega/k, t)|. \tag{3.7}$$

I. e.; at a point in the wave where $E > 0$ the number of electrons $f_{0e} + f_{1e}(\omega/k)$ must be larger than the number of ions, and when $E < 0$, the contrary applies. That is, the ions in mean over a wavelength are taking energy from the wave, while the electrons are giving energy to the wave, the latter quantity being larger than the former.

When $|f_{1e}(x, \omega/k, t)| = |f_{1i}(x, \omega/k, t)|$

we have a neutral electrostatic wave. This does not mean that nothing unusual takes place in the plasma. We have still the resonance phenomenon. Some particles always loose energy, other always gain energy and the linear theory is valid only a short line.

We also want to point out that the criterium (3.5) or equivalently (3.7) is not sufficient for instability as we do not know if the real dispersion relation has a solution such that ω/k satisfies this criterium.

IV. Drift Instabilities in the Presence of an External Magnetic Field

We shall now include an external magnetic field in our calculations and shall also limit ourselves to equilibrium distribution functions which are Maxwellian i. e.

$$f_{0i}(\varrho, v_z) = N \left(\frac{M}{2 \pi \varkappa T_i} \right)^{3/2} \exp\left\{ -\frac{M}{2 \varkappa T_i} (\varrho^2 + v_z^2) \right\}, \tag{4.1}$$

$$f_{0e}(\varrho, v_z) = N \left(\frac{m}{2 \pi \varkappa T_e} \right)^{3/2} \exp\left\{ -\frac{m}{2 \varkappa T_e} [\varrho^2 + (v_z - u)^2] \right\}. \tag{4.2}$$

Here $\varrho^2 = v_x^2 + v_y^2$, T_e and T_i are the electron and ion temperature respectively, $\varkappa$ = BOLTZMANN's constant.

The external magnetic field which is homogeneous is directed along the z-axis and we have introduced a drift u in the electron distribution function given by (4.2).

For the case that the ions are motionless and that $u = 0$, the absorption has been calculated in a previous paper [18], to be referred to as paper II. As the formulas there can be generalized to the present case without much additional calculations, we shall take advantage of this and refer the reader to that paper for details.

From paper II we find that the absorption eq. (17) is

$$A = \frac{\pi e^2}{\varkappa T k^2 k_2} \langle E_0^2 e^{-2\gamma t} \rangle \operatorname{Im} \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} f_{0e}(\varrho, v_z) \frac{(n \omega_c + k_2 v_z)^2 J_n^2(k_1 r)}{v_z - u_n} \varrho \, d\varrho \, dv_z \tag{4.3}$$

when $\gamma < 0$ (growing waves), and

$$A = \frac{\pi e^2}{\varkappa T k^2 k_2} \langle E_0^2 e^{-2\gamma t} \rangle \operatorname{Im} \sum_{-\infty}^{\infty} \int_{0}^{\infty} \varrho \, d\varrho \, J_n^2(k_1 r) \cdot \cdot \left[\int_{-\infty}^{\infty} \frac{f_0(\varrho, v_z) (n \omega_c + k_2 v_z)^2}{v_z - u_n} dv_z + 2 \pi i \omega'^2 f_0(\varrho, u_n) \right], \tag{4.4}$$

when $\gamma > 0$ (damped waves). Here an analytic continuation has been performed in order to make γ continuous for $\gamma \to \pm 0$.

The notation is: $k = |\boldsymbol{k}| =$ wave number, $k_1 = k \sin \theta$, $k_2 = k \cos \theta$ where $\theta =$ angle between $\mathbf{B}_0$ and $\boldsymbol{k}$, $\omega' = \omega - i \gamma$, $u_n = (\omega' - n \omega_c)/k_2$, $J_n(k_1 r)$ is the Bessel function of first kind and n'th order, $\omega_c =$ gyration frequency and $r = \varrho/\omega_c$ gyration radius for electrons. We have also used the fact that for a pure longitudinal wave

$$E_x = \frac{k_1}{k} E, \qquad E_z = \frac{k_2}{k} E, \text{ where } \quad \mathbf{E} = \mathbf{E}_0 \exp\{i(\boldsymbol{k}\,\boldsymbol{r} - \omega' t)\}. \; (\boldsymbol{r} = \text{position vector.}) \tag{4.5}$$

The perturbation of the distribution function is then given by (eq. II-13)

$$f_{1e} = \frac{e E}{\varkappa T_e k} f_{0e} \, e^{-i k_1 r \cos \varphi} \sum_{-\infty}^{\infty} i^{n+1} J_n(k_1 r) \frac{n \omega_{ce} + k_2 v_z}{n \omega_{ce} + k_2 v_{0z} - \omega'}. \tag{4.6}$$

Here f_{0e} is given by (4.2) with $u = 0$. When $u \neq 0$, the only difference is that the numerator $n \omega_{ce} + k_2 v_z$ must be replaced by $n \omega_{ce} + k_2 (v_z - u)$. The consequence for A is that $(n \omega_{ce} + k_2 v_z)^2$ must be replaced by $(n \omega_{ce} + k_2 v_z)$ $[n \omega_{ce} + k_2 (v_z - u)]$. The energy absorbed by the ions is found from (4.3) and (4.4) on replacing ω_{ce} by ω_{ci}, the ion cyclotron frequency, m by M and f_{0e} by f_{0i}.

Adding the resonant absorption due to ions and the absorption due to electrons and assuming $\gamma/k_2 \to 0$, we get for the total resonant absorption [either of the expressions (4.3) and (4.4) can be used for this case]:

$$A = \frac{E_0^2}{4 \pi} e^{-2\gamma t} \sqrt{\frac{\pi}{8}} \frac{\omega}{k^2 k_2} \sum_{-\infty}^{\infty} \left[\omega \, \omega_{pi}^2 \left(\frac{M}{\varkappa T_i} \right)^{3/2} e^{-\nu_i} I_n(\nu_i) \exp\left\{ - \frac{M}{2 \varkappa T_i} \left(\frac{\omega - n \omega_{ci}}{k_2} \right)^2 \right\} \right.$$
$$\left. + (\omega - k_2 u) \, \omega_{pe}^2 \left(\frac{m}{\varkappa T_e} \right)^{3/2} e^{-\nu_e} I_n(\nu_e) \exp\left\{ - \frac{m}{2 \varkappa T_e} \left(\frac{\omega - n \omega_{ce}}{k_2} - u \right)^2 \right\} \right]. \tag{4.7}$$

To arrive at this result, we have used the identity

$$\int_0^{\infty} J_n^2 \left(\frac{k_1}{\omega_{cj}} \varrho \right) \exp\left\{ - \frac{m_j}{2 \varkappa T_j} \varrho^2 \right\} \varrho \, d\varrho = \frac{\varkappa T_j}{m_j} e^{-\nu_j} I_n(\nu_j) \tag{4.8}$$

where $I_n(\nu_j)$ is the modified Bessel function of the first kind and where $\nu_j = k_1^2 \varkappa T_j / m_j \, \omega_{cj}^2$ (j = ion or electron).

From eq. (4.7) we are now able to derive the damping factor when the energy density W is known.

To find the criterium for instability we only have to demand $A < 0$, i. e. the expression in the parenthese [ ] must be negative. From this, the minimum current or drift u necessary to give instability can be found.

Different special cases may now be studied.

Case I. Let us assume a) $\dfrac{|\omega - k_2 u|}{k_2 v_e} \ll 1$, b) $\dfrac{\omega}{k_2 v_i} \gg 1$, c) $\dfrac{k_1 v_e}{\omega_{ce}} \ll 1$, d) $\dfrac{k_1 v_i}{\omega_{ci}} \ll 1$, (4.9, .10)

where $v_e = (2 \varkappa T_e/m)^{1/2}$ and $v_i = (2 \varkappa T_i/M)^{1/2}$.

Then only $I_0(\nu_e) \approx 1$ needs be kept in the series expansions and we obtain

$$A = \frac{E_0^2}{4\pi} e^{-2\gamma t} \sqrt{\frac{\pi}{8}} \frac{\omega}{k^2 k_2} (\omega - k_2 u)\, \omega_{\mathrm{pe}}^2 \left(\frac{m}{\varkappa T_{\mathrm{e}}}\right)^{3/2}. \tag{4.11}$$

The condition for instability becomes simply

$$u > \omega / k_2 . \tag{4.12}$$

As the dispersion equation for this case gives, (BERNSTEIN et al. [9])

$$\omega = \omega_{\mathrm{pi}} k_2 \lambda_{\mathrm{De}} \left[1 + (k_2 \lambda_{\mathrm{De}})^2 + \frac{T_{\mathrm{e}}}{2 T_{\mathrm{i}}} (k_1 R_{\mathrm{i}})^2\right]^{-1/2}, \qquad \lambda_{\mathrm{De}} = \left(\frac{\varkappa T_{\mathrm{e}}}{4 \pi N e^2}\right)^{1/2}$$

the condition can also be written

$$u > \omega_{\mathrm{pi}} \lambda_{\mathrm{De}} \left[1 + (k_2 \lambda_{\mathrm{De}})^2 + \frac{T_{\mathrm{e}}}{2 T_{\mathrm{i}}} (k_1 R_{\mathrm{i}})^2\right)^2\Big]^{-1/2}, \tag{4.13}$$

(R_{i} = ion gyration radius).

Case II. We shall now assume $\nu_e \ll 1$, which means that we can put $I_0(\nu_e) \approx 1$, $I_{n \neq 0} = 0$, when (4.9a) is retained. For the ions we assume $\omega \approx -\omega_{\mathrm{ci}}$ (ω_{ci} is negative) and that only the therm with $n = -1$ gives any essential contribution to the absorption:

$$\left|\frac{\omega + \omega_{\mathrm{ci}}}{k_2 v_{\mathrm{i}}}\right| \ll 1, \quad \left|\frac{\omega - n\,\omega_{\mathrm{ci}}}{k_2 v_{\mathrm{i}}}\right| \gg 1, \quad \text{for other } n. \tag{4.14}$$

Then we get:

$$A = \frac{E_0^2}{4\pi} e^{-2\gamma t} \sqrt{\frac{\pi}{8}} \frac{\omega}{k^2 k_2} \left[(\omega - k_2 u)\, \omega_{\mathrm{pe}}^2 \left(\frac{m}{\varkappa T_{\mathrm{e}}}\right)^{3/2} + \omega\, \omega_{\mathrm{pi}}^2 \left(\frac{M}{\varkappa T_{\mathrm{i}}}\right)^{3/2} e^{-\nu_{\mathrm{i}}} I_1(\nu_{\mathrm{i}})\right]. \tag{4.15}$$

Hence for instability $u > \dfrac{\omega}{k_2} \left[1 + \sqrt{\dfrac{M}{m}} \left(\dfrac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\right)^{3/2} e^{-\nu_{\mathrm{i}}} I_1(\nu_{\mathrm{i}})\right]$.

Case III. We assume $\omega \approx n_1 \omega_{\mathrm{ci}}$, $\nu_e \ll 1$, $\left|\dfrac{\omega - n_1 \omega_{\mathrm{ci}}}{k_2 v_{\mathrm{i}}}\right| \ll 1$, $\left|\dfrac{\omega - n\,\omega_{\mathrm{ci}}}{k_2 v_{\mathrm{i}}}\right| \gg 1$ for $n \neq n_1$ and (4.9a). (4.16)

Then

$$A = \frac{E_0^2}{4\pi} \sqrt{\frac{\pi}{8}} \frac{\omega}{k^2 k_2} \left[(\omega - k_2 u)\, \omega_{\mathrm{pe}}^2 \left(\frac{m}{\varkappa T_{\mathrm{e}}}\right)^{3/2} + \omega\, \omega_{\mathrm{pi}}^2 \left(\frac{M}{\varkappa T_{\mathrm{i}}}\right)^{3/2} e^{-\nu_{\mathrm{i}}} I_n(\nu_{\mathrm{i}})\right]. \tag{4.17}$$

If $\nu_{\mathrm{i}} \ll 1$, one may write $I_n(\nu_{\mathrm{i}}) = \nu_{\mathrm{i}}^n / n!\, 2^n$ and $e^{-\nu_{\mathrm{i}}} \approx 1$, which gives for instability

$$u > \frac{\omega}{k_2} \left(1 + \left(\frac{M}{m}\right)^{1/2} \left(\frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\right)^{3/2} \frac{\nu_{\mathrm{i}}^n}{2^n\, n!}\right). \tag{4.18}$$

V. Expression for the Current Density, Transverse to E and B_0

Instead of calculating more special cases of A we shall turn our interest to another effect closely related to the instability or stability treated above.

As mentionned already, the cause of the energy absorption is the particles which move with such a velocity that the electric field they experience has a constant component $\boldsymbol{E}'$. In the non-magnetic case these particles are moving with the velocity ω/k and in the magnetic case all particles with a v_z component equal to $(\omega - n\,\omega_c)/k_2$ experience such a constant field. We must then in the latter case also expect a constant drift $\boldsymbol{v}_{\mathrm{D}} = [\boldsymbol{E}' \times \boldsymbol{B}_0]/B_0^2$ for just these particles, being thus transverse to both the propagation vector $\boldsymbol{k}$ and the magnetic field. Since $\boldsymbol{E}'$, the electric field experienced by these resonance-particles, changes sign sinusoidaly along $\boldsymbol{k}$, the particles drift both along the positive y-axis as well as along the negative one. But since also the number of particles changes along the wave as mentionned in section 2, a net current may develop. This is the case for both electrons and ions. The particle drift may also be such that the electron current and the ion current cancel each other, and a zero current density results. This happens when the waves in the plasma have zero damping, i. e., for neutral waves.

As an adequat expression for the flux of resonant particles, we may calculate the current density

$(j_y)_{\text{res}}$. We shall also derive the relation with the damping factor and the absorption A, calculated in the previous section. We could as well calculate the transport of momentum or energy.

The drift velocity $\boldsymbol{v}_{\text{D}}$ is directed along the y-axis and has the value $v_{\text{D}} = -E_x/B_0$. The current density is then when averaged over one wave length λ:

$$(j_y)_{\text{res}} = \frac{e}{B_0} \int_{\text{res}} \overline{E_x(f_{1\text{e}} - f_{1\text{i}})}\, \mathrm{d}^3\boldsymbol{v}\,, \tag{5.1}$$

as the average of $E_x(f_{0\text{e}} - f_{0\text{i}})$ is zero. As we see $(j_y)_{\text{res}}$ is of second order in the perturbed quantities.

By modifying expression (4.6) we have

$$f_{1\text{e}} = \frac{e\,E}{\varkappa\, T_{\text{e}}\, k} f_{0\text{e}}\, e^{-i k_1 r \cos\varphi} \sum_{-\infty}^{\infty} i^{n+1} J_n(k_1 r_{\text{e}}) \frac{n\,\omega_{\text{ce}} + k_2(v_z - u)}{[n_{0\text{e}}]} e^{i n \varphi}\,, \tag{5.2}$$

$$f_{1\text{i}} = \frac{-e\,E}{\varkappa\, T_{\text{i}}\, k} f_{0\text{i}}\, e^{-i k_1 r \cos\varphi} \sum_{-\infty}^{\infty} i^{n+1} J_n(k_1 r_{\text{i}}) \frac{n\,\omega_{\text{ci}} + k_2 v_z}{[n_{0\text{i}}]} e^{i n \varphi}\,, \tag{5.3}$$

where $[n_{0\text{j}}] = n\,\omega_{\text{cj}} + k_2 v_z - \omega + i\gamma = n'_{0\text{j}} + i\gamma$, (say) and where the complex frequency $\omega' = \omega - i\gamma$, where ω is real.

By introducing (5.2) and (5.3) into (5.1) we obtain after integrating over φ and averaging in space:

$$(j_y)_{\text{res}} = \frac{\pi\, e^2 E_0^2 k_1}{B_0\, \varkappa\, k^2} e^{-2\gamma t}$$
$$\times \sum_{-\infty}^{\infty} \left\{ \int_0^{\infty} \varrho\, \mathrm{d}\varrho \int_{-\infty}^{\infty} \mathrm{d}v_z \left[\frac{f_{0\text{e}}}{T_{\text{e}}} \frac{\gamma[n\,\omega_{\text{ce}} + k_2(v_z - u)]}{n_{0\text{e}}'^2 + \gamma^2} J_n^2\left(\frac{k_1 \varrho}{\omega_{\text{ce}}}\right) + \frac{f_{0\text{i}}}{T_{\text{i}}} \frac{\gamma(n\,\omega_{\text{ci}} + k_2 v_z)}{n_{0\text{i}}'^2 + \gamma^2} J_n^2\left(\frac{k_1 \varrho}{\omega_{\text{ci}}}\right) \right] \right\}. \tag{5.4}$$

Here we have employed the relation

$$\int_0^{2\pi} i^n \exp[-i k_1 r \cos\varphi + i n \varphi]\, \mathrm{d}\varphi = 2\pi J_n(k_1 r)\,. \tag{5.5}$$

If we now assume that $\gamma/k_2 \to 0$ and further use the relation (4.8) we can perform the integrations over $\mathrm{d}v_z$ and $\varrho\,\mathrm{d}\varrho$. The result is, when (4.1) and (4.2) are inserted:

$$(j_y)_{\text{res}} = \frac{E_0^2 e^{-2\gamma t} k_1}{8\sqrt{2\pi}\, k^2 k_2} \sum_{-\infty}^{\infty} \left\{ (\omega - k_2 u)\, \omega_{\text{pe}}^2 \left(\frac{m}{\varkappa\, T_{\text{e}}}\right)^{3/2} e^{-\nu_{\text{e}}} I_n(\nu_{\text{e}}) \exp\left\{ -\frac{m}{2\varkappa\, T_{\text{e}}} \left[\left(\frac{\omega - n\,\omega_{\text{ce}}}{k_2}\right) - u \right]^2 \right\} \right.$$
$$\left. + \omega_{\text{pi}}^2\, \omega \left(\frac{M}{\varkappa\, T_{\text{i}}}\right)^{3/2} e^{-\nu_{\text{i}}} I_n(\nu_{\text{i}}) \exp\left\{ -\frac{M}{2\varkappa\, T_{\text{i}}} \left(\frac{\omega - n\,\omega_{\text{ci}}}{k_2}\right)^2 \right\} \right\}. \tag{5.6}$$

By taking account of the expression for the absorption eq. (4.7), the current $(j_y)_{\text{res}}$ can be written

$$(j_y)_{\text{res}} = \frac{k_1}{\omega\, B_0} A\,. \tag{5.7}$$

From the calculations we infer that the current $(j_y)_{\text{res}}$ is non-zero when $k_1 \neq 0$, i. e., when the wave propagation is not along the magnetic field, and when $A \neq 0$. We also observe that $(j_y)_{\text{res}} \neq 0$ whether the wave is a damping or growing one, that is whether the drift velocity u, is above or below the critical velocity u_{c} for instability. $(j_y)_{\text{res}}$ only changes sign when one passes from $u > u_{\text{c}}$ to $u < u_{\text{c}}$.

We also repeat that even when $A = 0$ and thus also $(j_y)_{\text{res}}$, we still have a flux of particles along the $\pm y$ directions.

We observe that only particles in a narrow region around $v_z' = (\omega - n\,\omega_{\text{c}})/k_2$ contributes to the current $(j_y)_{\text{res}}$. This is connected to the fact that only these particles have a constant drift velocity. However, other particles lying in a little larger region around v_z' will move very far before they return, and if the plasma is limited in the y-direction, this would mean an increased flux of resonant — and almost resonant particles along this direction.

At last we should like to mention that the derived current (5.7) could be measured if, by some means, the electric field E could be kept undamped even in a case where $A \neq 0$. If, however, the longitudinal oscillations are intrinsic waves in the plasma, the existence of an absorption different from zero leads to a damping of the electric field. The damping leads in turn to a polarization current, where all the particles in the main plasma, as distinguished from the resonant particles, take place. This current averaged over one wave-length, has the same magnitude as $(j_y)_{\text{res}}$ and the opposite direction, thus cancelling

Acknowledgments

The author is grateful to Dr. TROCHERIS and DIETRICH VOSLAMBER for stimulating discussions of the present paper.

The investigation has in part been supported by the Royal Norwegian Counsil for Scientific and Industrial Research.

Beschleunigung von Plasma

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(Z. Naturforschg. **18 a**, 453—459 [1963]; eingegangen am 17. Dezember 1962)

The generation and properties of plasmoids emitted by a conical electrodeless ring discharge are investigated. Image converter, magnetic probes, compensated magnetic loops and a retarding field analyzer were used for diagnostics.

There are two mechanisms of emission depending on the initial plasma density.

At low densities, when the ions can be regarded as free particles, the acceleration is caused only by the electromagnetic field at the beginning of the discharge.

At high densities, when the plasma behaves hydromagnetically, the emission mechanism is governed by trapped magnetic fields in the plasma. Under certain initial conditions a plasmoid with a trapped magnetic field is generated, which exists during a time of flight of the order of 10 μs.

Furthermore it is planned to accelerate this plasmoid by a special transmission line.

Es wird über Versuche berichtet, bei denen Plasma auf gerichtete (longitudinale) Energien im keV-Bereich beschleunigt wird. Durch teilweise Transformation dieser longitudinalen Energien in transversale soll ein Einfang in einer geeigneten Magnetfeldkonfiguration ermöglicht werden, so daß ein Plasma hoher Temperatur entsteht.

Am einfachsten sind diese Vorgänge, wenn man die Wechselwirkung der Ionen im Plasma vernachlässigen kann, d. h., wenn man das Plasma mit dem Einteilchenmodell beschreiben kann. Die Beschleunigung kann durch ein inhomogenes zeitlich schnell veränderliches Magnetfeld erfolgen, während sich zur Energietransformation Methoden anbieten, wie sie von FEDORCHENKO et al.[1], SINELNIKOV et al.[2] und DREICER et al.[3] für monoenergetische Elektronen verwendet wurden.

Ist die Wechselwirkung der Teilchen nicht mehr zu vernachlässigen, so daß sich das Plasma z. B. hydromagnetisch verhält, so werden die Vorgänge vor allem durch im Plasma eingefangene Felder komplexer. Auch in diesem Falle kann man inhomogene zeitlich schnell ansteigende Magnetfelder zur Beschleunigung verwenden. Die erreichbare Plasmaenergie ist jedoch durch den erforderlichen technischen Aufwand auf die Größenordnung 100 eV begrenzt. Höhere Energien kann man durch Verwendung eines beschleunigten magnetischen Kolbens mittels einer Laufzeitkette erreichen[4,5].

Diese Beschleunigung sowie die Energietransformation hängen von den Eigenschaften des Plasmas ab. Insbesondere beeinflussen eingefangene Felder Stabilität und Verluste während der Beschleunigung durch den magnetischen Kolben.

In dieser Arbeit wird untersucht, welche Eigenschaften ein durch eine elektrodenlose, konische Ringentladung erzeugtes Plasma in Abhängigkeit von der Druckverteilung des Gases zur Zeit der Zündung hat. Eine Beschleunigung des Plasmas mit Hilfe einer Laufzeitkette wird diskutiert.

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